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FLOATING POINT ERROR BOUND IN THE PRIME FACTOR FFT*

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ABSTRACT

The prime factor FFT (PF FFT), developed by Kolba and Parks, makes use of recent computational complexity results by Winograd to compute the DFT with a fewer number of multiplications than that required by the FFT. Patterson and McClellan have derived an expression for the MSE in the PF FFT assuming finite precision fixed point arithmetic. In this paper we derive a bound on the MSE in the PF FFT assuming floating point arithmetic. In the course of the derivation an expression for the actual MSE is also presented, but is seen to be too complicated to be of practical use.

I. INTRODUCTION

Fairly recently a new class of algorithms has emerged for computing the DFT with a fewer number of multiplications than that required by the FFT. The first of these algorithms was developed by Winograd [1,2] and makes use of his formulation for performing convolution with the minimum number of multiplications [3]. This algorithm has been termed the Winograd Fourier transform algorithm (WFTA) [4]. An unnested version of the WFTA has been proposed by Kolba and Parks and termed the prime factor FFT (PF FFT) [5].

It is of interest to investigate the effects of finite register length in these new algorithms. Patterson and McClellan have derived expressions for the average MSE in both the WFTA and PF FFT, assuming a statistical error model and fixed point arithmetic [6]. In this paper we restrict attention to the PF FFT and derive an expression for the MSE assuming floating point arithmetic. The resulting expression is quite cumbersome, but a bound on the MSE is also derived which is relatively easy to compute.

II. PRELIMINARIES

A. PF FFT Algorithm

A one dimensional Winograd type DFT algorithm can be represented in matrix notation as

$$\underline{Y} = CDA\,\underline{y} \tag{1}$$

where $\underline{y}$ is the input vector and $\underline{Y}$ is the DFT output. The rectangular matrices A and C correspond to the input and output stages of the algorithm and contain only 0 and ± 1 entries. The matrix D is diagonal and contains the only multiplications in the algorithm. As an example, the 3 point DFT may be written as [5]

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & -j\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} . \tag{2}$$

In this particular example A and C are square, but this is not generally the case.

An M-dimensional PF FFT is a cascade of M modules, each consisting of several of the computations given by (1). The PF FFT may be written as

$$\underline{X} = [C_M D_M A_M \times \cdots \times C_1 D_1 A_1]\underline{x} \tag{3}$$

where "x" denotes a Kronecker product, and $\underline{x}$ and $\underline{X}$ are the DFT input and output respectively. It is assumed that the dimension of $C_m D_m A_m$ is N_m and that the dimension of $\underline{x}$ is N with $N = \prod_{m=1}^{M} N_m$ and the N_m relatively prime. Thus to compute the DFT according to (3), only N_m point DFT's are required.

Fig. 1 gives a pictorial representation of (3) for M=2 modules. Each plane in the figure is a computation as given by (1). To date, algorithms have been published for only four mutually prime sequence lengths N_m. Hence, M=4 is the maximum number of modules.

B. Characterization of Floating Point Errors

We shall be concerned with binary machines using floating point arithmetic with a double precision accumulator. Thus, each machine number may be expressed as $(\text{sign})\cdot a \cdot 2^b$ where the mantissa 'a' is a fraction between $\frac{1}{2}$ and 1 and the exponent 'b' is an integer. It shall be assumed that B bits are used for the mantissa and that enough bits are alloted to the exponent to prevent overflow.

If we let $fl(\cdot)$ denote the machine number resulting from a real floating point operation, then it is well known that [7] for most machines

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$$fl(x+y) = (x+y)(1+\delta_1)$$
$$fl(x \cdot y) = (x \cdot y)(1+\delta_2) \quad (4)$$

where the errors δ_1 and δ_2 satisfy $-2^{-B}/2<\delta_i\leq 2^{-B}/2$ for rounding and $-2^{-B}<\delta_i\leq 0$ for truncation.

The errors δ_1 and δ_2 are typically modelled as random variables, independent of x and y and uniformly distributed on their range [7]. The bound to be derived in the next section will require knowledge of only the mean squared values of the δ_i, which can be easily computed. For example, assuming rounding we have for real addition and multiplication

$$E\{\delta_i^2\} = \frac{2^{-2B}}{12}$$

For the remainder of this paper we shall assume rounding arithmetic.

The PF FFT is composed of many short one dimensional DFT's, each implemented as in (1). In the next section we shall require an expression for the error vector at the output of a single one dimensional DFT. Let $\underline{y}$ and $\underline{Y}$ be the respective input and desired output of a single N_m point DFT. Then

$$\underline{Y} = C_m D_m A_m \underline{y}^R + jC_m D_m A_m \underline{y}^I \quad (5)$$

where $\underline{y}^R$ and $\underline{y}^I$ are the real and imaginary parts of $\underline{y}$ respectively. Denote the actual machine output by $\hat{\underline{Y}}$. Then the error vector at the DFT output, $\underline{v}(m) = \hat{\underline{Y}} - \underline{Y}$, may be written as

$$\underline{v}(m)= Q(m)\underline{y}^R + j\bar{Q}(m)\underline{y}^I + [G(m)\underline{Y}^R + j\bar{G}(m)\underline{Y}^I] \quad (6)$$

where Q(m) is an error matrix associated with a single N_m point DFT of a real input array. The error matrix Q(m) may be obtained as follows. The actual output $\hat{\underline{Y}}$, resulting from a real input array, is computed by substituting fl(·) for each multiplication and addition occuring in (1). The ith substitution is made with an error source δ_i where it is assumed the $\{\delta_i\}$ are uncorrelated. If desired, $d_{kk}(1+\delta_i)$ may be substituted for the diagonal elements d_{kk} in D, so that the error due to storage of the multiplier coefficients is also accounted for. All second order effects involving terms of the form $\delta_i\delta_j$ are dropped from $\hat{\underline{Y}}$. The output $\underline{Y}$ is then subtracted, giving $\underline{v}(m)$ to first order. Each element of $\underline{v}(m)$ is a linear combination of the y_i^R with coefficients (the entries in Q(m)) given by linear combinations of the δ_i. The matrix $\bar{Q}(m)$ is the error matrix associated with an N_m point DFT of a purely imaginary array. $\bar{Q}(m)$ has the same form as Q(m), however the elements in these two matrices are assumed to be uncorrelated since they arise from different multiplications. The last term in (6) is due to adding the DFT of $\underline{y}^R$ to the DFT of $j\underline{y}^I$ in (5). Both G(m) and $\bar{G}(m)$ are diagonal matrices with each element having variance $2^{-2B}/12$. A fact which will be neglected is that the first element in both G(m) and $\bar{G}(m)$ is actually zero. This follows since the component of Y_1 due to $\underline{y}^R$ is purely real and the component of Y_1 due to $j\underline{y}^I$ is purely imaginary so that no error occurs in adding these two components.

III. FLOATING POINT ERROR BOUND

It is desired to bound the average MSE (over all outputs) at the PF FFT output. This error results from computational errors which originate within each module in (3) and then propagate to the output. It is expedient to consider the effects of the errors from each module separately.

It is somewhat difficult to picture the PF FFT for M>2. However, each portion of the computation, from the mth module to the output, may still be represented in a manner similar to Fig. 1. Fig. 2 illustrates a particular stage $S_\ell(m)$, which is the transformation from a portion of the mth module input to the PF FFT output. The PF FFT contains $N_1 \cdots N_{m-1}$ of these stages in parallel. Hence, the index ℓ runs from 1 to $N_1 \cdots N_{m-1}$ and each stage computes only part of the PF FFT output.

Consider the kth N_m point DFT in stage $S_\ell(m)$. Let the error at the jth output of this DFT, due to errors originating within this DFT, be defined as

$$\epsilon_{jk}^{\ell}(m) \quad \begin{array}{l} j=1,\ldots,N_m \\ k=1,\ldots,(N_{m+1}\cdots N_M) \\ \ell=1,\ldots,(N_1\cdots N_{m-1}) \\ m=1,\ldots,M \end{array}$$

Then let

$$e_{rs}^{\ell}(m) \quad \begin{array}{l} r=1,\ldots,(N_{m+1}\cdots N_M) \\ s=1,\ldots,N_m \\ \ell=1,\ldots,(N_1\cdots N_{m-1}) \\ m=1,\ldots,M \end{array}$$

be the rth output of the sth $(N_{m+1}\cdots N_M)$ point DFT in $S_\ell(m)$ due to the $\epsilon_{jk}^{\ell}(m)$.

We shall first compute

$$\frac{1}{N}\sum_{\ell}\sum_{r}\sum_{s} E\,|\,e_{rs}^{\ell}(m)|^2$$

which is the average MSE at the PF FFT output due to errors originating within the mth module. For fixed ℓ and s it follows from Parseval that

$$\sum_r E|e_{rs}^{\ell}(m)|^2 = (N_{m+1}\cdots N_M)\sum_r E|\epsilon_{sr}^{\ell}(m)|^2 .$$

Therefore

$$\frac{1}{N}\sum_{\ell}\sum_{r}\sum_{s} E|e_{rs}^{\ell}(m)|^2 = (N_{m+1}\cdots N_M)\sigma^2(m) \quad (7)$$

where

$$\sigma^2(m) \triangleq \frac{1}{N}\sum_{\ell}\sum_{r}\sum_{s} E|\epsilon_{sr}^{\ell}(m)|^2 \quad (8)$$

is the average MSE at the output of the mth module due to errors originating within that module.

An expression for $\sigma^2(m)$ is now required in terms of the algorithm parameters. To investigate the errors occuring within the mth module, (3) is rewritten using a standard Kronecker product identity giving

$$\underline{X} = [(C_M D_M A_M)\times\cdots\times(C_{m+1}D_{m+1}A_{m+1})\times I_m\times\cdots\times I_1]\cdot$$
$$[I_M\times\cdots\times I_{m+1}\times(C_m D_m A_m)\times I_{m-1}\times\cdots\times I_1]\cdot \tag{9}$$
$$[I_M\times\cdots\times I_m\times(C_{m-1}D_{m-1}A_{m-1})\times\cdots\times(C_1D_1A_1)]\underline{x}.$$

Similarly $\underline{X}$ may be written as a product of M matrices times $\underline{x}$ where the ith matrix represents the transformation performed by the ith module. In (9) we have broken the computation around the mth module. The output $\underline{x}(m)$ of this module may be written as

$$\underline{x}(m) = [I_M\times\cdots\times I_{m+1}\times(C_m D_m A_m)\times I_{m-1}\times\cdots\times I_1]\underline{x}(m-1) \tag{10}$$

where $\underline{x}(m-1)$ is the input to the mth module given by $\underline{x}(m-1)=[I_M\times\cdots\times I_m\times(C_{m-1}D_{m-1}A_{m-1})\times\cdots\times(C_1D_1A_1)]\underline{x}$.

It is apparent from (6) and (10) that the error $\underline{\epsilon}(m)$ in $\underline{x}(m)$ may be expressed as

$$\underline{\epsilon}(m) = P(m)\underline{x}^R(m-1) + j\bar{P}(m)\underline{x}^I(m-1) + F(m)\underline{x}^R(m) + j\bar{F}(m)\underline{x}^I(m) \tag{11}$$

where

$$P(m) \triangleq [I_M\times\cdots\times I_{m+1}\times Q(m)\times I_{m-1}\times\cdots\times I_1]^{ss}$$
$$F(m) \triangleq [I_M\times\cdots\times I_{m+1}\times G(m)\times I_{m-1}\times\cdots\times I_1]^{ss}. \tag{12}$$

Here $[\cdot]^{ss}$ indicates that each time Q(m) or G(m) is repeated in the Kronecker product, its elements are superscripted relating them to their respective $C_m D_m A_m$ implementation. Without this additional superscript, the δ_i arising from one implementation of $C_m D_m A_m$ would be assumed to be identical to those arising from another implementation. However, since the inputs to the various implementations are different, we shall in fact assume these errors to be uncorrelated.

From the definition of $\underline{\epsilon}(m)$, the quantity $\sigma^2(m)$, given by (8) may be written as

$$\sigma^2(m) = \frac{1}{N} E\{\underline{\epsilon}^*(m)\,\underline{\epsilon}(m)\}$$

where "*" denotes complex conjugate transpose. Substituting from (11) gives

$$\begin{aligned}\sigma^2(m) = \frac{1}{N} E\{ & \underline{x}^{R*}(m-1)P^*(m)P(m)\underline{x}^R(m-1)\\ & + \underline{x}^{I*}(m-1)\bar{P}^*(m)\bar{P}(m)\underline{x}^I(m-1)\\ & + \underline{x}^{R*}(m)F^*(m)F(m)\underline{x}^R(m)\\ & + \underline{x}^{I*}(m)\bar{F}^*(m)\bar{F}(m)\underline{x}^I(m)\}\end{aligned} \tag{13}$$

where all cross terms are zero and have been omitted since P(m), $\bar{P}$(m), F(m), and $\bar{F}$(m) are all uncorrelated and zero-mean. The last two terms in (13) may be further simplified since

$$\begin{aligned}E\{\underline{x}^{R*}(m)F^*(m)F(m)\underline{x}^R(m)\} &= E\{\underline{x}^{R*}(m)E\{F^*(m)F(m)\}\underline{x}^R(m)\}\\ &= E\{\underline{x}^{R*}(m)\frac{2^{-2B}}{12}\, I\, \underline{x}^R(m)\}\\ &= \frac{2^{-2B}}{12} E\{\underline{x}^{R*}(m)\underline{x}^R(m)\}\ .\end{aligned}$$

The second equality follows because F(m) is a diagonal matrix with each element having variance $2^{-2B}/12$. The last term in (13) is equal to a similar expression so that the sum of the last two terms is given by

$$\frac{2^{-2B}}{12}[E\{\underline{x}^{R*}(m)\underline{x}^R(m)\} + E\{\underline{x}^{I*}(m)\underline{x}^I(m)\}] = \frac{2^{-2B}}{12} E\{\underline{x}^*(m)\underline{x}(m)\}.$$

$\sigma^2(m)$ given by (13) can now be rewritten as

$$\begin{aligned}\sigma^2(m) = \frac{1}{N} E\{ & \underline{x}^{R*}(m-1)P^*(m)P(m)\underline{x}^R(m-1)\\ & + \underline{x}^{I*}(m-1)\bar{P}^*(m)\bar{P}(m)\underline{x}^I(m-1)\\ & + \frac{2^{-2B}}{12}\underline{x}^*(m)\underline{x}(m)\}\ .\end{aligned} \tag{14}$$

So far, only errors originating within the mth module have been considered. We shall now formulate an expression for the total average MSE at the PF FFT output due to all modules. It may be seen from (11) and (12) that the error vector at the output of the mth module depends on Q(m) and G(m). However, Q(m) and G(m) are uncorrelated with Q(i) and G(i) for i ≠ m and hence $\underline{\epsilon}(m)$ is uncorrelated with $\underline{\epsilon}(i)$. The total average MSE at the PF FFT output is therefore the sum of the average MSE's due to each module. From (7) the total average MSE σ^2 is given by

$$\begin{aligned}\sigma^2 &= \frac{1}{N}\sum_m\sum_\ell\sum_r\sum_s E|e^\ell_{rs}(m)|^2\\ &= [\sum_{m=1}^{M-1}\sigma^2(m)\prod_{i=m+1}^{M} N_i] + \sigma^2(M).\end{aligned} \tag{15}$$

We may then substitute (14) for $\sigma^2(m)$, however the first two terms in this quantity will be difficult to compute. As an alternative we shall now derive a bound on $\sigma^2(m)$ which is much easier to compute than $\sigma^2(m)$ itself.

The first term in (14) may be bounded as follows. Letting $T \triangleq P^*(m)P(m)$ gives

$$\begin{aligned}\underline{x}^{R*}(m-1)P^*(m)P(m)\underline{x}^R(m-1) &= \sum_k x^R_k(m-1)\sum_\ell t_{k\ell}\, x^R_\ell(m-1)\\ &= |\sum_k\sum_\ell x^R_k(m-1)t_{k\ell}x^R_\ell(m-1)|\\ &\le \sum_k\sum_\ell |x^R_k(m-1)||t_{k\ell}|\cdot|x^R_\ell(m-1)|\end{aligned}$$

$$\le \sum_k \sum_\ell \mid x_k^R(m-1) \mid \sqrt{t_{kk}} \sqrt{t_{\ell\ell}} |x_\ell^R(m-1)|$$

$$= (\sum_k |x_k^R(m-1)| \sqrt{t_{kk}})^2$$

$$\le \sum_k |x_k^R(m-1)|^2 \sum_\ell t_{\ell\ell}$$

$$= \sum_k |x_k^R(m-1)|^2 \sum_\ell \sum_n \mid p_{\ell n}(m) \mid^2$$

where the second inequality follows since T is non-negative definite, and the last inequality follows by Cauchy-Schwartz. The second term in (14) can be bounded in an identical fashion giving

$$\sigma^2(m) \le \frac{1}{N} E\{ \sum_k |x_k^R(m-1)|^2 \sum_\ell \sum_n E|p_{\ell n}(m)|^2 + \sum_k \mid x_k^I(m-1) \mid^2 \sum_\ell \sum_n E \mid \bar{p}_{\ell n}(m) \mid^2 + \frac{2^{-2B}}{12} \underline{x}^*(m)\underline{x}(m) \}. \quad (16)$$

Consider $\Sigma \Sigma E |p_{\ell n}(m)|^2$ in (16). It is apparent from (12) that the matrix P(m) contains the elements of Q(m) repeated N/N_m times, each time appropriately superscripted as mentioned before. All other entries in P(m) are zeroes. Regardless of the superscripts we have that

$$\sum_{\ell=1}^{N} \sum_{n=1}^{N} E|p_{\ell n}(m)|^2 = \frac{N}{N_m} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} E|q_{ij}(m)|^2 \quad (17)$$

Using (16), (17), and the fact that $E|\bar{q}_{ij}(m)|^2 = E|q_{ij}(m)|^2$ gives

$$\sigma^2(m) \le \frac{1}{N} E\{ \frac{N}{N_m} \sum_{n=1}^{N} |x_n(m-1)|^2 \sum_{k=1}^{N_m} \sum_{\ell=1}^{N_m} E|q_{k\ell}(m)|^2 + \frac{2^{-2B}}{12} \sum_{n=1}^{N} |x_n(m)|^2 \}.$$

Now, if $x_n(m)$ is the nth output of the mth module it follows from Parseval that

$$\sum_{n=1}^{N} |x_n(m)|^2 = N_m \sum_{n=1}^{N} |x_n(m-1)|^2 .$$

By induction then

$$\sigma^2(m) \le \sum_{n=1}^{N} E|x_n|^2 [\frac{N_1 \cdots N_{m-1}}{N_m} \sum_{k=1}^{N_m} \sum_{\ell=1}^{N_m} E|q_{k\ell}(m)|^2 + \frac{N_1 \cdots N_m}{N} \frac{2^{-2B}}{12}]$$

where $x_n = x_n(0)$ is the nth input to the PF FFT. Finally, substituting this into (15) gives the desired bound

$$\sigma^2 \le \sum_{n=1}^{N} E|x_n|^2 [\sum_{m=1}^{M} ((\frac{N}{N_m^2} \sum_{k=1}^{N_m} \sum_{\ell=1}^{N_m} E|q_{k\ell}(m)|^2) + \frac{2^{-2B}}{12})].$$

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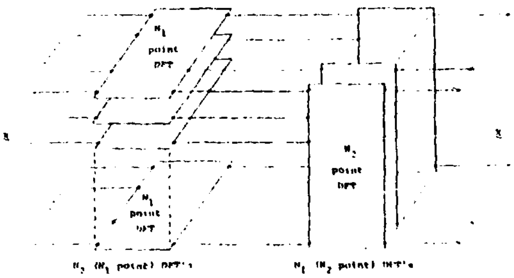

Fig. 1. Example PF FFT of length $N = N_1 \cdot N_2$.

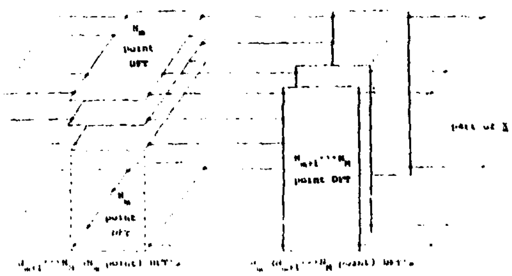

Fig. 2 ... stage ... portion of the computation from the mth module input to the PF FFT output